

Capacity as a Fundamental Metric for Mechanism Design in the Information Economy

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Abstract. The auction theory literature has so far focused mostly on the design of mechanisms that takes the revenue or the efficiency as a yardstick. However, scenarios where the *capacity*, which we define as “*the number of bidders the auctioneer wants to have a positive probability of getting the item*”, is a fundamental concern are ubiquitous in the information economy. For instance, in sponsored search auctions (SSA’s) or in online ad-exchanges, the true value of an ad-slot for an advertiser is inherently derived from the conversion-rate, which in turn depends on whether the advertiser actually obtained the ad-slot or not; thus, unless the capacity of the underlying auction is large, key parameters, such as true valuations and advertiser-specific conversion rates, will remain unknown or uncertain leading to inherent inefficiencies in the system. In general, the same holds true for all information goods/digital goods. We initiate a study of mechanisms, which take capacity as a yardstick, in addition to revenue/efficiency. We show that in the case of a single indivisible item one simple way to incorporate capacity constraints is via designing mechanisms to sell probability distributions, and that under certain conditions, such optimal probability distributions could be identified using a Linear programming approach. We define a quantity called *price of capacity* to capture the tradeoff between capacity and revenue/efficiency. We also study the case of sponsored search auctions. Finally, we discuss how general such an approach via probability spikes can be made, and potential directions for future investigations.

1 Introduction

“The tension between giving away your information- to let people know what you have to offer- and charging them for it to recover your costs is a fundamental problem in the information economy.”

— Carl Shapiro and Hal R. Varian in *Information Rules: A Strategic Guide to the Network Economy*, Harvard Business School Press (1998).

The advent of the Internet as a big playground for resource sharing among selfish agents with diverse interests, and the emergence of Web as a giant platform for hosting information has raised a plethora of opportunities for commerce, as well as, a plenty of new design, pricing and complexity problems. One good example of a multi-billion dollar industry evolved as a consequence of Web is the sponsored search advertising

(SSA), making fortunes for Internet Search giants such as Google and Yahoo!, and has got tremendous attention in academia recently, due to various interesting research problems originated as a result of this continuously growing industry[1, 3, 7, 12, 13, 15, 17–20, 24, 25]. One of the most important concern for such an industry or in general for the information economy is the “*pricing problem*”. For example, for the goods like an ad-slot in the SSA which has no intrinsic value and is perishable, it is not clear what price should it be sold for. Similarly, for a digital good, where the cost of reproduction is negligible, the standard way of pricing based on the production cost does not work. Therefore, auctions are becoming a popular pricing mechanism in electronic commerce as they automatically adjust prices to market conditions, and specifically prices gets adjusted according to its value to the consumers rather than to the production costs. Auction theory has a pretty impressive literature [10, 14]- from the lovely Vickrey auction to the sponsored search auctions[7, 24] and the auctions of digital goods[8, 9]. The literature has so far focused mostly on the design of mechanisms that takes the revenue or the efficiency as a yardstick. This is perfectly logical as these are two very important metrics from the viewpoint of the seller/auctioneer and the society respectively. However, the scenarios where the *capacity*, which we define as “*the number of bidders the auctioneer wants to have a positive probability of getting the item*”, is a fundamental concern are ubiquitous in the information economy¹. For instance, in the sponsored search auctions or in online ad-exchanges, the true value of an ad-slot for an advertiser is inherently derived from the conversion-rate, which in turn depends on whether the advertiser actually obtained the ad-slot or not, which in turn depends on the capacity. In general, the same holds true for all information goods/digital goods.

In the present paper, our goal is to first motivate *capacity* as a fundamental metric in designing auctions in the information economy and then to initiate study of such a design framework for some simple and interesting scenarios. In Section 2, we motivate the capacity as a fundamental and interesting additional metric on the top of revenue/efficiency for mechanism design. In Section 3, we start with the *capacity* constrained framework for selling an indivisible item. We propose a simple way to incorporate capacity constraints via designing mechanisms to sell probability distributions. We show that such optimal probability distributions could be identified using a Linear programming approach, objective being revenue, efficiency or a related function. Further, we define a quantity called *price of capacity* to capture the tradeoff between capacity and revenue/efficiency and derive upper bounds on it. In Section 4, we discuss the case of sponsored search auctions and also note that the auctioneer controlled probability spikes based auctions suggests a new model for sponsored search advertising, where a click is sold directly and not indirectly via allocating impressions. In Section 5, we conclude with a list of research directions for future work, inspired by the present paper.

2 Motivation: the need for an additional metric

Experience goods: A bidder might not know her true valuation for the item unless she acquires it sometimes meaning that the true value is inherently derived from the actual

¹ Clearly, this capacity can be increased indefinitely albeit sometimes at the cost of revenue.

acquisition of the item. Such a good is called an *experience good*[16, 22, 2]. Experience goods are ubiquitous in the information economy as clearly all information goods are experience goods[22, 2]. Sometimes, a particular good might also act as an experience for another good. For example, a particular song from a singer might act as an experience for another song of that singer. Therefore, in the auction of an experience good, the values for some of the bidders can be known to them as they might have experienced it from an earlier purchase of this or a related item, and for the other bidders the values are still unknown and they have to simply guess this value if they are participating in the auction. For the second kind of bidders, even if their values might turn out to be pretty high, their guesses might not be high enough to actually acquire the item when revenue/efficiency is the only goal. Moreover, they might be *loss-averse*[23] and would not bid a high value at all, due to the potential risk involved if the item does not turn out to be of high value to them. Therefore, it is important that such bidders be given a chance to acquire the item, and consequently capacity becomes a fundamental metric in designing the mechanisms to achieve this goal.

Two-fold exploration in sponsored search auctions: First, the ad-slots in the SSA are necessarily experience goods for a new advertiser and the estimates for the advertisers getting lower ranked slots is also generally poor as they hardly get any clicks. The value of an ad-slot is derived from the clicks themselves (i.e. rate of conversion or purchase given a click), and therefore, unless the bidder actually obtains a slot and receives user clicks, there is essentially no means for her to estimate her true value for the associated keyword.

Second, even if all the true valuations are known to the corresponding bidders, for each bidder the SSA involves a parameter called *quality score* of the bidder which is defined as the expected clickability of the bidder for the associated keyword if she obtains a slot. This parameter is also not known a priori and the auctioneer needs to estimate it. Certainly, a model that automatically allows one to estimate these key parameters (i.e. Click-Through-Rates and true values) is desirable. Indeed, some mechanisms to incorporate explorations for estimating such important parameters has started to appear in literature[18, 25]. Capacity as an additional metric can provide a generic framework for designing such exploration based mechanisms.

Avoiding over-exposure in online advertising: Typically, the online ad-exchanges such as Right Media or DoubleClick convince their advertisers that their ads will not be over exposed to users. One way of avoiding such an over-exposure could be via increasing capacity.

Uncertainty and switching costs: Let us consider a production company H buying raw materials (multiple units of a good) from providers A, B and C via a reverse-auction and H is uncertain about the time these providers might take to deliver the raw material to H . The providers are very likely to lie about the delivery times and it is hard to incorporate delivery times in designing the auction. If the goal of the auctioneer (i.e. the company H) is cost minimization, it will buy the raw material from the provider with the minimum ask (assume that the ask of B is smaller than that of A which is smaller than that of C). Now if the provider B lied about the delivery time at least

for a significant fraction of the total required units, H 's production gets delayed. If H wants to switch to some other provider, run another auction and buys from A , now it will buy at a higher cost and there is still a delay in H 's production as A will take its time in delivery too. The time taken by A could actually be smaller than that of B 's for the remaining units, however, still there is a delay. Moreover, such delay might persist further as A could also lie about its delivery time. It might have been better if H would buy not only from B to start with and give A, C a chance as well. Therefore, one way to reduce such delay times could be via increasing *capacity* as per our definition.

3 Selling an indivisible item

3.1 The model to sell via probability spikes

There is a single *indivisible* item for sale. There are N bidders interested in the item. The bidder i has a value v_i for this item². The item is sold via an auction on an experiment designed by the seller/auctioneer. The experiment has M outcomes (O_1, O_2, \dots, O_M) with associated probabilities (p_1, p_2, \dots, p_M) , where $\sum_{i=1}^M p_i = 1, p_i \geq 0 \forall i$. Therefore, the item is essentially sold via an auction of the probability spikes (p_1, p_2, \dots, p_M) wherein the auctioneer can choose these probability spikes in advance or adaptively based on bidders' reports so as to achieve some defined goal such as maximizing her profit or efficiency or to accommodate a wider pool of bidders. Bidders bid on the experiment by reporting bids b_i 's to indicate their respective values of the item. *At most one probability spike is assigned to each bidder*. Thus there are effectively two steps in this auction model.

- Stage 1 (*commit/compete*): The bidders report their bids b_i 's and by way of using some mechanism, the auctioneer assigns the probability spikes to them and decides corresponding payments to be made by them. Let us call a bidder a *prospective winner* if she was assigned one of the probability spikes.
- Stage 2 (*win or lose*): The experiment is performed. If the outcome of the experiment is O_j , then the *prospective winner* assigned to the spike p_j is declared the *winner*, and is given the item.

Further, the auctioneer could choose various payment schemes such as -

- *Betting*: Every *prospective winner* is charged its payment decided in *compete/commit* stage irrespective of whether she will be a *winner* or not.
- *Pay-per-acquisition*: A bidder is charged the amount decided in *compete stage* only when she is a *winner* i.e. only when she actually acquires the item.

The above model can also be interpreted as selling of a single *divisible* item in terms of specified fractional bundles, the bundles corresponding to the probability spikes.

² Note that, as discussed earlier in Section 2, for some bidders v_i 's are the actual true values while for some others these are just crude estimates/guesses.

3.2 Mechanisms to sell probability spikes

Without loss of generality, let us assume that $p_1 \geq p_2 \geq \dots \geq p_M \geq 0$. Further, for notational simplicity let $\mathcal{M} = \{1, 2, \dots, M\}$, $\mathcal{N} = \{1, 2, \dots, N\}$. Let $\sigma : \mathcal{N} \rightarrow \mathcal{N}$ be the allocation rule and $h : \mathcal{N} \rightarrow \mathbb{R}_+$ be the payment rule decided in *compete stage*. Thus, for $j \in \mathcal{M}$, the spike j is assigned to the bidder $\sigma(j)$ and for $j \in \mathcal{N} - \mathcal{M}$, the bidder $\sigma(j)$ is not assigned any spike. Further, for $j \in \mathcal{M}$, h_j is the expected payment to be made by the bidder $\sigma(j)$ and $h_j = 0$ otherwise. Therefore, the expected utility of the bidder assigned to spike j is given by $u_{\sigma(j)} = p_j v_{\sigma(j)} - h_j$ for $j \in \mathcal{M}$ and is zero otherwise. For the sake of simplicity, let $p_1 > p_2 > \dots > p_M > 0$, then the famous VCG mechanism ranks the bidders by their bids (and true values v_i 's as being a truthful mechanism) and charges them their respective opportunity costs. That is, $\sigma(j)$ is the bidder with the j th maximum bid and $h_j = \sum_{i=j}^{M-1} (p_i - p_{i+1}) v_{\sigma(i+1)} + p_M v_{\sigma(M+1)}$ for $j \in \mathcal{M}$ and zero otherwise. If the payment is done via the *betting* model then this is the amount charged to bidder j in the *compete/commit* stage. If the payment is done via *pay-per-acquisition* and j is the *winner* then she is charged an amount $\frac{h_j}{p_j}$, and therefore, her expected payment is still h_j as p_j is the probability that she wins. Therefore, the auctioneer's revenue is

$$R_{VCG} = \sum_{i=1}^{M-1} (p_i - p_{i+1}) i v_{\sigma(i+1)} + p_M M v_{\sigma(M+1)}. \quad (1)$$

Let $\theta_j = p_j - p_{j+1}$; $j = 1, 2, \dots, M-1$ & $\theta_M = p_M$ be the spike gaps, then $p_j = \sum_{i=j}^M \theta_i$. The condition $\sum_{i=1}^M p_i = 1$ translates to $\sum_{i=1}^M i \theta_i = 1$ and $p_1 \geq p_2 \geq \dots \geq p_M \geq 0$ translates to $\theta_j \geq 0 \forall j = 1, 2, \dots, M$. Therefore,

$$R_{VCG} = \sum_{i=1}^M \theta_i i d_i \text{ where } d_i = v_{\sigma(i+1)} \quad (2)$$

and clearly $d_i \geq d_{i+1}$ as VCG ranks by true values.

Lemma 1 *The revenue of the auctioneer in the VCG mechanism for selling probability spikes (p_1, p_2, \dots, p_M) can be expressed as $R_{VCG} = \sum_{i=1}^M \theta_i i d_i$ where $\theta_i = p_i - p_{i+1}$ for all $i = 1, 2, \dots, M-1$, $\theta_M = p_M$ and $d_i \geq d_{i+1}$ for all $i = 1, 2, \dots, M$.*

Further, the efficiency for the VCG mechanism is

$$\begin{aligned} E_{VCG} &= \sum_{j=1}^M p_j v_{\sigma(j)} = \sum_{j=1}^M \left(\sum_{i=j}^M \theta_i \right) v_{\sigma(j)} \\ &= \sum_{i=1}^M \left(\sum_{j=1}^i v_{\sigma(j)} \right) \theta_i = \sum_{i=1}^M \theta_i i d_i \end{aligned} \quad (3)$$

$$\text{where } d_i = \frac{1}{i} \sum_{j=1}^i v_{\sigma(j)}. \quad (4)$$

Further, we have

$$\begin{aligned}
d_i - d_{i+1} &= \frac{1}{i} \sum_{j=1}^i v_{\sigma(j)} - \frac{1}{(i+1)} \sum_{j=1}^{i+1} v_{\sigma(j)} = \frac{1}{i(i+1)} \left\{ \sum_{j=1}^i (i+1)v_{\sigma(j)} - \sum_{j=1}^{i+1} i v_{\sigma(j)} \right\} \\
&= \frac{1}{i(i+1)} \left\{ \sum_{j=1}^i v_{\sigma(j)} - i v_{\sigma(i+1)} \right\} = \frac{1}{i(i+1)} \left\{ \sum_{j=1}^i (v_{\sigma(j)} - v_{\sigma(i+1)}) \right\} \\
&\geq 0 \text{ as } v_{\sigma(j)} \geq v_{\sigma(i+1)} \forall j = 1, 2, \dots, i.
\end{aligned}$$

Lemma 2 *The efficiency in the VCG mechanism for selling probability spikes (p_1, p_2, \dots, p_M) can be expressed as $E_{VCG} = \sum_{i=1}^M \theta_i d_i$ where $\theta_i = p_i - p_{i+1}$ for all $i = 1, 2, \dots, M-1$, $\theta_M = p_M$ and $d_i \geq d_{i+1}$ for all $i = 1, 2, \dots, M$.*

Definition 1. *We say that a linear function H of spike-gaps θ_j 's is gap-wise monotone if $H = \sum_{j=1}^M \theta_j j d_j$, where d_j 's do not depend on gaps θ_j 's and $d_j \geq d_{j+1}$ for all $j = 1, 2, \dots, M$.*

Definition 2. *A mechanism for selling probability spikes is called gap-wise monotone if the revenue of the auctioneer at the prescribed equilibrium point is gap-wise monotone and is called strongly gap-wise monotone if the social value (i.e. efficiency) at the prescribed equilibrium point is gap-wise monotone as well.*

Therefore, from Lemma 1 and Lemma 2 we obtain the following theorem.

Theorem 3 *The VCG mechanism for selling probability spikes is strongly gap-wise monotone.*

Define, $u_i^*(h) = \max\{\max_{j \in \mathcal{M}} (p_j v_i - h_j), 0\}$.

Definition 3. Walrasian Equilibrium: *Let σ be an allocation and h be a payment rule, then (σ, h) is called a Walrasian equilibrium if for all $j \in \mathcal{N}$, $u_{\sigma(j)} = u_{\sigma(j)}^*(h)$.*

Following [21, 6, 4], it is not hard to establish the following lemma.

Lemma 4 *Let σ be an allocation and h be a payment rule, then (σ, h) is a Walrasian equilibrium iff it is efficient.*

Therefore, at a Walrasian equilibrium, bidders are ranked according to their values and efficiency can be written as in the case of VCG i.e. $\sum_{j=1}^M p_j v_{\sigma(j)}$ where $v_{\sigma(j)} \geq v_{\sigma(j+1)}$ for all $j = 1, 2, \dots, M$.

Theorem 5 *Let (σ, h) be a Walrasian equilibrium for selling probability spikes then the efficiency at this equilibrium is gap-wise monotone.*

This means that optimal efficiency is always gap-wise monotone. Further, the optimal omniscient auction (i.e. when the auctioneer knows everyone's true value v_i 's) extracts a revenue equal to $\sum_{j=1}^M p_j v_{\sigma(j)}$, where $v_{\sigma(j)} \geq v_{\sigma(j+1)}$ for all $j = 1, 2, \dots, M-1$. Therefore, the optimal revenue of omniscient auction is also gap-wise monotone.

3.3 A Generic Framework for Selecting Optimal Spikes

In this section, we develop a Linear Programming approach to identify optimal probability spikes subject to the capacity constraints in terms of spikes gaps, where the objective is a gap-wise monotone function. For such functions, it is simpler to put the constraints in terms of spike-gaps than in terms of spikes themselves, however, it won't be hard to see that a similar approach can also be developed if we put the constraints in terms of the spikes, as well as, in the case of functions more general than the gap-wise monotone. For the sake of simplicity we omit any such details.

Let H be a *gap-wise monotone* function and $\{\epsilon_j\}_{j=1}^M$ be a generic set of parameters with the property that

$$\sum_{i=1}^M i\epsilon_i \leq 1. \quad (5)$$

$$\epsilon_j \geq 0; j = 1, 2, \dots, M \quad (6)$$

and let us consider the following Linear Programming Problem in variables θ_j 's,

$$\text{Max } H = \sum_{j=1}^M \theta_j j d_j$$

$$\text{s.t. } \sum_{i=1}^M i\theta_i = 1 \quad (7)$$

$$\theta_j \geq \epsilon_j; j = 1, 2, \dots, M \quad (8)$$

The dual problem is

$$\begin{aligned} \text{Min } x_0 - \sum_{j=1}^M \epsilon_j x_j \\ \text{s.t. } x_j \geq 0; j = 1, 2, \dots, M \\ -j d_j + j x_0 - x_j = 0; j = 1, 2, \dots, M \end{aligned} \quad (9)$$

and the *KKT* conditions are

$$\begin{aligned} \sum_{i=1}^M i\theta_i &= 1 \\ \theta_j &\geq \epsilon_j; j = 1, 2, \dots, M \\ x_j &\geq 0; j = 1, 2, \dots, M \\ -j d_j + j x_0 - x_j &= 0; j = 1, 2, \dots, M \\ x_j(\epsilon_j - \theta_j) &= 0; j = 1, 2, \dots, M \end{aligned} \quad (10)$$

and therefore an optimal solution is

$$\begin{aligned} x_0^* &= d_1 \\ x_j^* &= j(d_1 - d_j) \end{aligned}$$

$$\begin{aligned}\theta_j^* &= \epsilon_j \forall j = 2, \dots, M \\ \theta_1^* &= 1 - \sum_{i=2}^M i\epsilon_i\end{aligned}\tag{11}$$

as it can be checked to satisfy the *KKT* conditions. The optimal value is

$$H^{OPT}(\{\epsilon_j\}) = d_1 - \sum_{j=2}^M j\epsilon_j(d_1 - d_j).\tag{12}$$

Clearly, the maximum of the optimal solution $H^{OPT}(\epsilon_1, \epsilon_2, \dots, \epsilon_M)$ over parameters $\{\epsilon_j\}$'s is attained when $\epsilon_j = 0$ for all $j = 1, 2, \dots, M$ and in that case

$$H^{OPT} = d_1.\tag{13}$$

Now, note that the primal optimal variables θ_j 's do not depend on the quantities d_j 's at all. Therefore, so long as H is *gap-wise monotone*, the optimal solution to the primal remains the same as in equation 11. It is quite intuitive as the best possible spike allowed by the capacity constraints is assigned to the best possible bidder (i.e. $\sigma(1)$), and all other spikes are the minimum possible as per the constraints.

Theorem 6 *Let $H = \sum_{j=1}^M \theta_j j d_j$ be a gap-wise monotone function and the spike-gaps θ_j 's satisfy conditions 8, 5 and 6, then the optimal choice of spike-gaps are given by equation 11 and the optimal value of H is given by equation 12.*

3.4 The Price of Capacity

Given parameters $\{\epsilon_j\}$, let us define the *capacity* as

$$\kappa(\{\epsilon_j\}) = \max_j \{j : \epsilon_j > 0\}.\tag{14}$$

Now consider the parameters $\{\tilde{\epsilon}_j\}$ satisfying the properties 5 and 6 such that $\kappa(\{\tilde{\epsilon}_j\}) = \kappa(\{\epsilon_j\}) + 1$. Given a gap-wise monotone function $H = \sum_{j=1}^M \theta_j j d_j$, we claim that such $\tilde{\epsilon}_j$'s satisfying properties 5 and 6 can always be obtained from ϵ_j 's such that $H^{OPT}(\{\tilde{\epsilon}_j\}) \geq H^{OPT}(\{\epsilon_j\})$ as long as $H^{OPT}(\{\epsilon_j\}) < H^{OPT}$, meaning that the *capacity* can always be increased without any loss in optimal value as long as we do not shoot over the absolute optimum H^{OPT} . We have,

$$\begin{aligned}H^{OPT}(\{\tilde{\epsilon}_j\}) - H^{OPT}(\{\epsilon_j\}) &= \sum_{j=2}^{\kappa(\{\epsilon_j\})} j\epsilon_j(d_1 - d_j) - \sum_{j=2}^{\kappa(\{\epsilon_j\})+1} j\tilde{\epsilon}_j(d_1 - d_j) \\ &= \sum_{j=2}^{\kappa(\{\epsilon_j\})} j(\epsilon_j - \tilde{\epsilon}_j)(d_1 - d_j) - (\kappa(\{\epsilon_j\}) + 1)\tilde{\epsilon}_{\kappa(\{\epsilon_j\})+1}(d_1 - d_{\kappa(\{\epsilon_j\})+1}).\end{aligned}$$

Now we can always choose $\tilde{\epsilon}_j$'s satisfying properties 5 and 6 by taking suitable $\tilde{\epsilon}_j \leq \epsilon_j; j = 1, \dots, \kappa(\{\epsilon_j\}) - 1, \tilde{\epsilon}_{\kappa(\{\epsilon_j\})} < \epsilon_{\kappa(\{\epsilon_j\})}$ and $\tilde{\epsilon}_{\kappa(\{\epsilon_j\})+1} \leq \frac{\sum_{j=2}^{\kappa(\{\epsilon_j\})} j(\epsilon_j - \tilde{\epsilon}_j)(d_1 - d_j)}{(\kappa(\{\epsilon_j\}) + 1)(d_1 - d_{\kappa(\{\epsilon_j\})+1})}$.

In particular, taking $\tilde{\epsilon}_j = \epsilon_j; j = 1, \dots, \kappa(\{\epsilon_j\}) - 1, \tilde{\epsilon}_{\kappa(\{\epsilon_j\})} < \epsilon_{\kappa(\{\epsilon_j\})}$ and $\tilde{\epsilon}_{\kappa(\{\epsilon_j\})+1} \leq \min \left\{ \frac{\sum_{j=2}^{\kappa(\{\epsilon_j\})} j(\epsilon_j - \tilde{\epsilon}_j)(d_1 - d_j)}{(\kappa(\{\epsilon_j\}) + 1)(d_1 - d_{\kappa(\{\epsilon_j\})+1})}, \frac{\kappa(\{\epsilon_j\})(\epsilon_{\kappa(\{\epsilon_j\})} - \tilde{\epsilon}_{\kappa(\{\epsilon_j\})})}{(\kappa(\{\epsilon_j\}) + 1)} \right\}$ does the job. An interesting

case to consider is when $\epsilon_j = \epsilon > 0$ for all $j \leq m$ and $\epsilon_j = 0$ otherwise. And in this case we can increase the capacity without loss in optimal value by taking $\tilde{\epsilon}_j = \tilde{\epsilon}$ and $\tilde{\epsilon}_j = 0$ otherwise, where

$$\tilde{\epsilon} \leq \min \left\{ \frac{2}{(m+1)(m+2)}, \frac{\sum_{j=2}^m j(d_1 - d_j)}{\sum_{j=2}^{m+1} j(d_1 - d_j)} \epsilon \right\}. \quad (15)$$

Now let us define

$$a := \min_j j : d_1 > d_j$$

then it is clear that

$$\begin{aligned} H^{OPT}(\{\epsilon_j\}) &= H^{OPT} \text{ whenever } \kappa(\{\epsilon_j\}) < a \\ &< H^{OPT} \text{ whenever } \kappa(\{\epsilon_j\}) \geq a \end{aligned}$$

and therefore, as long as $\kappa(\{\epsilon_j\}) \geq a$ capacity can always be increased without any loss in the optimal value as discussed above, however there is a strict decrease in the optimal value if we wish to increase capacity from $a-1$ to a . We can naturally define a parameter which we call *price of capacity* as follows:

$$\nu(\{d_j\}) := \max_{\{\epsilon_j\} : \kappa(\{\epsilon_j\}) = a} \left(\frac{H^{OPT}}{H^{OPT}(\{\epsilon_j\})} \right) \quad (16)$$

$$\begin{aligned} &= \max_{\{\epsilon_j\} : \kappa(\{\epsilon_j\}) = a} \left(\frac{d_1}{d_1 - a\epsilon_a(d_1 - d_a)} \right) \\ &\leq \frac{d_1}{d_a} = \frac{d_{a-1}}{d_a} \end{aligned} \quad (17)$$

Thus, *price of capacity* is the worst possible loss in optimal value while increasing capacity from $a-1$ to a . Again let us consider the case when all non-zero $\epsilon_j = \epsilon$, then

$$\begin{aligned} \nu(\{d_j\}) &= \max_{0 < \epsilon \leq \frac{2}{a(a+1)}} \left(\frac{d_1}{d_1 - a\epsilon(d_1 - d_a)} \right) = \frac{d_1}{d_1 - \frac{2}{a+1}(d_1 - d_a)} \\ &= \frac{(a+1)}{(a-1) + 2\left(\frac{d_a}{d_1}\right)} \leq 1 + \frac{2}{a-1} \leq 3 \end{aligned} \quad (18)$$

Often our goal will be to maximize efficiency or revenue subject to the capacity constraints, and consequently such a loss may not be considered good. Therefore, its really a price that we are paying for increasing capacity.

4 Sponsored Search Auctions

As we discussed in the Section 2, one nice motivation for the study of capacity as a metric for mechanism design comes from the sponsored search advertising. We first

describe the formal SSA model. Formally, in the current models, there are K slots to be allocated among N ($\geq K$) bidders (i.e. the advertisers). A bidder i has a true valuation v_i (known only to the bidder i) for the specific keyword and she bids b_i . The expected *click through rate* (CTR) of an ad put by bidder i when allocated slot j has the form $CTR_{i,j} = \gamma_j e_i$ i.e. separable in to a position effect and an advertiser effect. γ_j 's can be interpreted as the probability that an ad will be noticed when put in slot j and it is assumed that $\gamma_j > \gamma_{j+1}$ for all $1 \leq j \leq K$ and $\gamma_j = 0$ for $j > K$. e_i can be interpreted as the probability that an ad put by bidder i will be clicked on if noticed and is referred to as the *relevance*(quality score) of bidder i . The payoff/utility of bidder i when given slot j at a price of p per-click is given by $e_i \gamma_j (v_i - p)$. As of now, Google as well as Yahoo! use schemes closely modeled as RBR(rank by revenue) with GSP(generalized second pricing). The bidders are ranked in the decreasing order of $e_i b_i$ and the slots are allocated as per this ranks. Let the $\sigma(i)$ be the bidder allocated to the slot i according to this ranking rule, then $\sigma(i)$ is charged an amount equal to $\frac{e_{\sigma(i+1)} b_{\sigma(i+1)}}{e_{\sigma(i)}}$ per-click. This mechanism has been extensively studied in recent years[7, 12, 24, 13, 3]. The solution concept that is widely adopted to study this auction game is a refinement of Nash equilibrium called *symmetric Nash equilibria(SNE)* independently proposed by Varian[24] and Edelman et al[7]. For notational simplicity, let $s_{\sigma(j+1)} = v_{\sigma(j+1)} e_{\sigma(j+1)}$, then under this refinement, the revenue of the auctioneer at equilibrium is given by

$$\sum_{i=1}^K (\gamma_j - \gamma_{j+1}) j s_{\sigma(j+1)}. \quad (19)$$

In this section, we discuss how to incorporate the capacity constraints in the keyword auctions being currently used by Google and Yahoo!. We understand that there could be several ways for doing so, however, we consider a very simple and intuitive way of incorporating the capacity constraints via probability spikes as follows:

- The first $K - 1$ slots are sold as usual to the $K - 1$ high-ranked bidders.
- The last slot (i.e. the K th slot) is sold via probability spikes (p_1, p_2, \dots, p_M) among the M bidders ranked K through $K + M - 1$. M is chosen to accommodate as many more bidders as the auctioneer wants.
- There is a single combined auction for both of the above.

Clearly, the single combined auction is equivalent to the keyword auction with $(K + M - 1)$ slots with position based CTRs taken as $(\gamma_1, \gamma_2, \dots, \gamma_{K-1}, \gamma_K p_1, \gamma_K p_2, \dots, \gamma_K p_M)$ i.e. $\tilde{\gamma}_i = \gamma_i$ for $i \leq K - 1$, $\tilde{\gamma}_{K+j-1} = \gamma_K p_j$ for $1 \leq j \leq M$ and $\tilde{\gamma}_i = 0$ otherwise. The revenue of the auctioneer at SNE is

$$\begin{aligned} R &= \sum_{j=1}^{K+M-1} (\tilde{\gamma}_j - \tilde{\gamma}_{j+1}) j s_{\sigma(j+1)} \\ &= \sum_{j=1}^{K-2} (\gamma_j - \gamma_{j+1}) j s_{\sigma(j+1)} + \gamma_{K-1} (K-1) s_{\sigma(K)} - \gamma_K p_1 (K-1) s_{\sigma(K)} \\ &\quad + \gamma_K \sum_{j=K}^{K+M-1} (p_{j+1-K} - p_{j+2-K}) j s_{\sigma(j+1)}. \end{aligned}$$

Now, maximizing R as a function of p_j 's is equivalent to maximizing the function

$$\begin{aligned} H &= -\gamma_K p_1 (K-1) s_{\sigma(K)} + \gamma_K \sum_{j=K}^{K+M-1} (p_{j+1-K} - p_{j+2-K}) j s_{\sigma(j+1)} \\ &= \sum_{j=1}^M \theta_j \{ (K+j-1) s_{\sigma(K+j)} - (K-1) s_{\sigma(K)} \} \text{ where } \theta_j = p_j - p_{j+1}; j = 1, 2, \dots, M-1 \text{ \& } \theta_M = p_M \\ &= \sum_{j=1}^M \theta_j j d_j \text{ where } d_j = \frac{1}{j} \{ (K+j-1) s_{\sigma(K+j)} - (K-1) s_{\sigma(K)} \}. \end{aligned}$$

Note that H may not be gap-wise monotone, however, it is not hard to see that the similar linear programming analysis as in the Section 3.3 can be done to compute the price of capacity in the present scenario of keyword auctions as well. We omit the details.

Selling clicks via auctioneer-controlled probability spikes: a new model for sponsored search advertising: The design framework in Section 3, suggests a new model for SSA, where the clicks are sold directly and not indirectly via allocating impressions. In the usual pay-per-click model, a click is assigned to one of the advertisers who has been allocated an impression in the page currently being viewed by the user; thus, the user's collective experience determines the probability that an impression winner would be the beneficiary of the click, and having set up the impressions and the user experience, the auctioneer does not actively control the probability with which a click will be allocated to a bidder. In the new model, a click could be considered as the indivisible item being sold via probability spikes. Instead of putting ads directly in the slots, the auctioneer could put some categories/information related to the specific keyword as a link. An auction based on probability spikes is run whenever a user clicks on this link and the user is directly taken to the landing page of the winning advertiser.

5 Future Work

In the present paper, our main goal was to motivate *capacity* as a fundamental metric in designing auctions in the information economy and then to initiate study of such a design framework for some simple and interesting scenarios such as single indivisible item and sponsored search advertising. However, there are myriad of other interesting scenarios where the capacity-enabled framework should be interesting to study. For example, the auctions of digital goods[8, 9], combinatorial auctions [5] for selling information goods in bundles, double auctions and ad-exchanges[11] etc. Further, probably the most important question that remains to be addressed is to identify the best way of putting capacity constraints and to see how generic the approach via probability spikes could be made. We can consider the most general case of selling any set of items e.g. heterogeneous, homogeneous, indivisible, or divisible or any combination their of. Let $\Sigma = \{\sigma_i\}$ be the set of all possible allocations. Further, the elements of Σ are named such that $\sigma_i \succeq \sigma_{i+1}$, where \succeq is auctioneer's preference over allocations. Then, these set of items can be sold with capacity constraints via probability spikes p_i 's with $p_i \geq p_{i+1}$, wherein σ_i is enforced with probability p_i .

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